

**NUMBER AND PHASE:
COMPLEMENTARITY AND JOINT MEASUREMENT UNCERTAINTIES**

PEKKA LAHTI, JUHA-PEKKA PELLONPÄÄ, AND JUSSI SCHULTZ

ABSTRACT. We show that number and canonical phase (of a single mode optical field) are complementary observables. We also bound the measurement uncertainty region for their approximate joint measurements.

1. INTRODUCTION

Analogously to position and momentum of a quantum object, number and phase of a single mode optical field are often considered as an example of a pair of observables which is complementary and for which the uncertainty relations put severe limitations both for preparations and measurements. However, since there is no phase shift covariant spectral measure solution to the quantum phase problem it has remained a challenge to formulate the exact content of these intuitive ideas for this pair of observables.

The notion of complementarity, which goes back to the 1927 Como lecture of Niels Bohr [1] and which was strongly advocated also by Wolfgang Pauli [2], is often discussed only rather vaguely and mostly in connection with Werner Heisenberg's uncertainty relations [3]. However, the notion of mutual exclusiveness which is associated with the idea of complementarity has rather straightforward independent formulations in quantum mechanics, and, like uncertainty, it has both probabilistic and measurement theoretical aspects. Along with Bohr [4], we say that two observables are complementary if all the instruments (measurements) which allow their unambiguous definitions are mutually exclusive. The notion of mutual exclusiveness of measurements is easily expressed with respect to the order structure of the set of quantum effects, sharp or unsharp. Following Pauli [2], one may also say that two observables are probabilistically complementary if certain predictions concerning the measurement outcomes of these observables are mutually exclusive. In addition, with the notion of value complementarity of two observables one often refers to the case where sharply defined value (exact knowledge) of one observable implies uniform distribution (complete ignorance) on the values of the other observable. These notions have obvious expressions in terms of the measurement outcome

probabilities of quantum mechanics. Straightforward formulations of the three versions of complementarity have been proposed and studied, for instance, in [5, 6].

Concerning number and phase, it is, perhaps, well known that they are probabilistically complementary as well as value complementary, see, for instance [7, Proposition 16.2 and 16.3], but it has remained an open question if among the phase shift covariant phase observables there is any which would be complementary with the number [8]. This question is now settled in Section 3 where it is shown that the canonical phase and number form a complementary pair.

Complementary observables are necessarily incompatible, that is, they cannot be measured jointly. This leads one to study their approximate joint measurements, a topic which has gained a substantial clarification in recent years. Rather than digging in the extensive history of the topic, we refer to the relevant chapters of the monograph [7]. In Section 4 we follow the ideas and methods initiated in [10, 11] and further developed, for instance, in [12, 13, 14], to bound the measurement uncertainty domain of the complementary pair of number and canonical phase.

Throughout the paper we use freely the standard notions and terminology of Hilbert space quantum mechanics. Yet, we start with a short account of the main terminology and the basic results concerning the canonical phase observable.

2. BASIC NOTIONS

Let \mathcal{H} be a Hilbert space, $\{|n\rangle \mid n \in \mathbb{N}\}$ an orthonormal basis of \mathcal{H} , and $N = \sum_{n=0}^{\infty} n|n\rangle\langle n|$ the corresponding number operator. Let $\mathcal{L}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ denote, respectively, the sets of bounded and trace class operators on \mathcal{H} . We also let $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ denote the set of positive, trace one operators (states). We denote by $\mathbf{N} : 2^{\mathbb{N}} \rightarrow \mathcal{L}(\mathcal{H})$ the spectral measure of N and call it the *number observable*. With any observable, like \mathbf{N} , we let \mathbf{N}_{ρ} denote the probability measure $Y \mapsto \mathbf{N}_{\rho}(Y) = \text{tr}[\rho \mathbf{N}(Y)]$ defined by the observable and a state $\rho \in \mathcal{S}(\mathcal{H})$.

Let $\mathcal{B}([0, 2\pi))$ be the Borel sigma algebra of $[0, 2\pi)$. By a *phase observable* we mean any normalized positive operator measure (semispectral measure) $\mathbf{E} : \mathcal{B}([0, 2\pi)) \rightarrow \mathcal{L}(\mathcal{H})$ which is covariant under the phase shifts generated by the number observable, that is, satisfies the condition $e^{i\theta N} \mathbf{E}(X) e^{-i\theta N} = \mathbf{E}(X \dot{+} \theta)$ for all $\theta \in [0, 2\pi)$ and $X \in \mathcal{B}([0, 2\pi))$, where $\dot{+}$ denotes addition modulo 2π . The structure of such observables is completely known, see, for instance, [15, 16, 7]. Among them there is the one referred to the *canonical* phase observable, which we denote by $\Phi : \mathcal{B}([0, 2\pi)) \rightarrow \mathcal{L}(\mathcal{H})$ and which has the effects

$$(2.1) \quad \Phi(X) = \sum_{m,n=0}^{\infty} \int_X e^{i(m-n)\theta} \frac{d\theta}{2\pi} |m\rangle\langle n|.$$

There are several properties which distinguish Φ as the *canonical* phase among all the phase observables \mathbf{E} . Without entering the whole list of such properties,¹ we mention here only the fact that, up to unitary equivalence, the canonical phase is the only phase observable which generates number shifts: $V^{(k)}\mathbf{N}(Y+k)(V^{(k)})^* = \mathbf{N}(Y)$, where $V^{(k)} = \int_0^{2\pi} e^{ik\theta} d\Phi(\theta)$ are the cyclic moment operators of Φ . We recall also that the spectrum of the effect $\Phi(X)$, $0 \neq \Phi(X) \neq I$, is the whole interval $[0, 1]$ with no eigenvalues. In particular, for any $\theta \in [0, 2\pi)$ and for any $\epsilon > 0$, the (operator) norm of the effect $\Phi((\theta - \epsilon, \theta + \epsilon) \cap [0, 2\pi))$ equals one. Thus, for each point $\theta \in [0, 2\pi)$ there is a sequence of unit vectors $(\psi_i)_{i \in \mathbb{N}}$ such that the probability measures $X \mapsto \langle \psi_i | \Phi(X) \psi_i \rangle$ tend, with increasing i , to the point measure δ_θ at θ . In such a case, the number probabilities $|\langle \psi_i | n \rangle|^2$ tend to zero for all n . Observing, in addition, that in the number states $|n\rangle$ the phase distribution is uniform, $\langle n | \Phi(X) | n \rangle = \int_X \frac{d\theta}{2\pi} = \ell(X)$, the probabilistic and the value complementarity of the pair (\mathbf{N}, Φ) become obvious.

As well-known, number \mathbf{N} and phase Φ are incompatible observables, that is, they cannot be measured jointly. Indeed, since \mathbf{N} is a spectral measure, their joint measurement \mathbf{M} would necessarily be of the product form, that is, $\mathbf{M}(n, X) = |n\rangle\langle n| \Phi(X) = \Phi(X) |n\rangle\langle n|$ for any $n \in \mathbb{N}, X \in \mathcal{B}([0, 2\pi))$ (see, for instance, [7, Proposition 4.8]). But this would imply that $\Phi(X) = \ell(X) I$, which contradicts (2.1).

Though Φ and \mathbf{N} have no joint observable, there are observables $\mathbf{M} : \mathcal{B}([0, 2\pi) \times \mathbb{N}) \rightarrow \mathcal{L}(\mathcal{H})$ having either Φ or \mathbf{N} as a margin, that is, either $\mathbf{M}_1 = \Phi$, with $\mathbf{M}_1(X) = \mathbf{M}(X \times \mathbb{N})$, or $\mathbf{M}_2 = \mathbf{N}$, with $\mathbf{M}_2(Y) = \mathbf{M}([0, 2\pi) \times Y)$. In either case the joint observable is a smearing of the exact margin. Indeed, if $\mathbf{N} = \mathbf{M}_2$, then $\mathbf{M}(X \times Y) = \mathbf{M}_1(X) \mathbf{N}(Y)$ (cf. above) and each $\mathbf{M}_1(X)$ is a function of N so that $\mathbf{M}(X \times Y) = \sum_{n \in Y} p(X, n) |n\rangle\langle n|$, with a Markov kernel $\mathcal{B}([0, 2\pi)) \times \mathbb{N} \ni (X, n) \mapsto p(X, n) \in [0, 1]$. On the other hand, if $\mathbf{M}_1 = \Phi$, then again there is a kernel $p : [0, 2\pi) \times 2^{\mathbb{N}} \rightarrow [0, 1]$ such that \mathbf{M} is obtained as

$$(2.2) \quad \mathbf{M}(X \times Y) = \int_X p(\theta, Y) d\Phi(\theta),^2$$

so that, in particular, for each $Y \in 2^{\mathbb{N}}$, $\mathbf{M}_2(Y) = \int_0^{2\pi} p(\theta, Y) d\Phi(\theta)$. The structural similarity of the two cases is due to the fact that both Φ and \mathbf{N} are rank-1 observables, for details, see [18, 19].

¹A reader interested in those properties of Φ may check the list of 19 items of [16, Sect. 4.8] together with some further properties [17, 18].

²We recall that this integral simply means that for each state ρ , $\text{tr}[\rho \mathbf{M}(X \times Y)] = \int_X p(\theta, Y) d\Phi_\rho(\theta)$, the integral of the (measurable) function $\theta \mapsto p(\theta, Y)$ with respect to the probability measure Φ_ρ .

The above results contain also the following well-known facts. In any of the sequential measurements (in either order), if the first measurement is exact, that is, measures either \mathbf{N} or Φ , then any information on the other observable coded in the initial state of the measured system is completely lost in the following precise sense: if, say, \mathbf{N} is measured first, with an instrument \mathcal{I} , in a state ρ , then the subsequent phase probabilities are $\text{tr}[\mathcal{I}(\mathbf{N})(\rho)\Phi(X)] = \text{tr}[\rho\mathcal{I}(\mathbf{N})^*(\Phi(X))]$, where the ‘distorted’ phase effects $\mathcal{I}(\mathbf{N})^*(\Phi(X))$ are smearings of the number observable for some kernel $n \mapsto p(X, n)$. Similarly, if one first performs an exact phase measurement, with an instrument \mathcal{J} , say, then the subsequent number probabilities are $\text{tr}[\mathcal{J}([0, 2\pi])(\rho)|n\rangle\langle n|] = \text{tr}[\rho\mathcal{J}([0, 2\pi])^*(|n\rangle\langle n|)]$, where the ‘distorted’ number effects $\mathcal{J}([0, 2\pi])^*(|n\rangle\langle n|)$ are smearings of the phase observable Φ with a kernel $\theta \mapsto p(\theta, \{n\})$.

We now turn to study the complementarity of the number and the canonical phase.

3. COMPLEMENTARITY OF THE PAIR (\mathbf{N}, Φ)

As already pointed out, the pair (\mathbf{N}, Φ) is known to be both probabilistically complementary and value complementary, but it has remained an open question if they are also complementary. This question will now be settled with Theorem 1 which shows that for each finite subset $Y \subset \mathbb{N}$ and $X \in \mathcal{B}([0, 2\pi))$, for which $\Phi(X) \neq I$, the greatest lower bound of the effects $\Phi(X)$ and $\mathbf{N}(Y)$ exists in the partially ordered set of effects $\mathcal{E}(\mathcal{H}) = \{E \in \mathcal{L}(\mathcal{H}) \mid 0 \leq E \leq I\}$ and equals the null effect, that is

$$(3.1) \quad \Phi(X) \wedge \mathbf{N}(Y) = 0.$$

It is this relation which we take to express the complementarity of the pair (\mathbf{N}, Φ) in the sense that all the measurements which serve to define these observables are mutually exclusive. In fact, if (3.1) were not true, then for some such X and Y there would be an effect E below both $\mathbf{N}(Y)$ and $\Phi(X)$, so that, in any state ρ , the probability $\text{tr}[\rho E]$ would also be a common lower bound for the corresponding number and the phase probabilities. Thus, with measuring the effect E in any state one would also get information from the effects $\mathbf{N}(Y)$ and $\Phi(X)$ in that state. Relation (3.1) excludes such measurements.

The order structure of the set of effects is known to be quite complicated when compared with the order structure of the set of projections. However, a characterization of pairs of effects $E, F \in \mathcal{E}(\mathcal{H})$ for which $E \wedge F$ exists has been obtained [20], and, in particular, it is known that if one of them is a projection then their greatest lower bound always exists [20, Corollary 3.1]. Therefore, $\Phi(X) \wedge \mathbf{N}(Y)$ exists for any $X \in \mathcal{B}(\mathbb{T})$ and $Y \subset \mathbb{N}$, and it remains to be shown that

all these meets are zero whenever Y is a finite set and X such that $\ell(X) < 1$ (i.e. $\Phi(X) \neq I$). Clearly, such a result depends on the explicit properties of the number and the canonical phase.

From now on we identify the phase interval $[0, 2\pi)$ (addition modulo 2π) with the torus \mathbb{T} in the usual way through the map $\theta \mapsto e^{i\theta}$, denoting still by $d\ell(\theta) = \frac{d\theta}{2\pi}$ the normalized measure on \mathbb{T} . Let \mathbf{Q} be the canonical spectral measure of the Hilbert space $\tilde{\mathcal{H}} = L^2(\mathbb{T})$ and let $\{e_k \mid k \in \mathbb{Z}\}$ be its Fourier basis, that is, $e_k(\theta) = e^{-ik\theta}$. Let $P_{\mathbb{N}}$ be the projection $\sum_{n=0}^{\infty} |e_n\rangle\langle e_n|$. The Naimark projection of \mathbf{Q} onto $P_{\mathbb{N}}(\tilde{\mathcal{H}})$, that is, the map $X \mapsto P_{\mathbb{N}}\mathbf{Q}(X)|_{P_{\mathbb{N}}(\tilde{\mathcal{H}})}$ is exactly of the form (2.1). In fact, \mathbf{Q} is the minimal Naimark dilation of Φ [7, Theorem 8.1].

We identify \mathcal{H} with the subspace $P_{\mathbb{N}}(\tilde{\mathcal{H}})$ of $\tilde{\mathcal{H}}$ via the isometry $V : |n\rangle \mapsto e_n$, so that $P_{\mathbb{N}} = VV^*$ and

$$\Phi(X) = V^*\mathbf{Q}(X)V = V^*P_{\mathbb{N}}\mathbf{Q}(X)V$$

for all $X \in \mathcal{B}(\mathbb{T})$.

Remark 1. Let \mathbf{P} be the spectral measure with the (atomic) projections $|e_k\rangle\langle e_k|$, $k \in \mathbb{Z}$. In [21, Example 4.2] it was shown that the pair (\mathbf{Q}, \mathbf{P}) of $L^2(\mathbb{T})$ is complementary, that is, $\mathbf{Q}(X) \wedge \mathbf{P}(Y) = 0$ for all $X \in \mathcal{B}(\mathbb{T})$, for which $\mathbf{Q}(X) \neq I_{\tilde{\mathcal{H}}}$, and for all finite $Y \subset \mathbb{Z}$. The corresponding result for the position-momentum pair (\mathbf{Q}, \mathbf{P}) of $L^2(\mathbb{R})$ is well known, see, e.g., [7, Proposition 8.2]. Though $\Phi(X) = V^*\mathbf{Q}(X)V$ and $\mathbf{N}(Y) = V^*\mathbf{P}(Y)V (= \mathbf{P}(Y))$, for $Y \subset \mathbb{N}$, the noncommutativity of $P_{\mathbb{N}}$ and $\mathbf{Q}(X)$ prevents one to conclude the disjointness of the effects $\Phi(X)$ and $\mathbf{N}(Y)$ directly from the disjointness of the projections $\mathbf{Q}(X)$ and $\mathbf{P}(Y)$.

Lemma 1. Let $\alpha \geq 0$ and $X \in \mathcal{B}(\mathbb{T})$ such that $\Phi(X) \neq I$. Then $\alpha|0\rangle\langle 0| \leq \Phi(X)$ implies $\alpha = 0$.

Proof. Suppose that $\alpha|0\rangle\langle 0| \leq \Phi(X) = V^*\mathbf{Q}(X)V$, that is, $\alpha|e_0\rangle\langle e_0| \leq V\Phi(X)V^* = P_{\mathbb{N}}\mathbf{Q}(X)P_{\mathbb{N}} = [\mathbf{Q}(X)P_{\mathbb{N}}]^*[\mathbf{Q}(X)P_{\mathbb{N}}]$ and note that $\Phi(X) \neq I$ if and only if $\mathbf{Q}(X) \neq I_{\tilde{\mathcal{H}}}$ if and only if $\ell(X) < 1$. Let $\mathcal{K} = \overline{\mathbf{Q}(X)P_{\mathbb{N}}(\tilde{\mathcal{H}})}$. Define an operator $D \in \mathcal{L}(\tilde{\mathcal{H}})$ by $D(\mathbf{Q}(X)\psi) = \sqrt{\alpha}\langle e_0|\psi\rangle e_0$, $\psi \in P_{\mathbb{N}}(\tilde{\mathcal{H}})$, and $D\varphi = 0$, $\varphi \in \mathcal{K}^\perp$. Indeed, D is clearly linear and well defined since, if $\mathbf{Q}(X)\psi = \mathbf{Q}(X)\psi'$, $\psi, \psi' \in P_{\mathbb{N}}(\tilde{\mathcal{H}})$, i.e. $\mathbf{Q}(X)\psi_- = 0$, $\psi_- = \psi - \psi'$, then

$$0 \leq \|D(\mathbf{Q}(X)\psi_-)\|^2 = \langle \psi_- | \alpha e_0 \rangle \langle e_0 | \psi_- \rangle \leq \langle \psi_- | P_{\mathbb{N}}\mathbf{Q}(X)P_{\mathbb{N}}\psi_- \rangle = \langle \psi_- | \mathbf{Q}(X)\psi_- \rangle = 0$$

so that $D(\mathbf{Q}(X)\psi) = D(\mathbf{Q}(X)\psi')$. Similarly, $\|D(\mathbf{Q}(X)\psi)\| \leq \|\mathbf{Q}(X)\psi\|$, $\psi \in P_{\mathbb{N}}(\tilde{\mathcal{H}})$, showing that D is bounded and thus extends to the whole $\tilde{\mathcal{H}}$. Since the range of D is $\mathbb{C}e_0$, one has $D = |e_0\rangle\langle \eta|$ for some $\eta \in \tilde{\mathcal{H}}$. In addition, since $D\mathbf{Q}(X)P_{\mathbb{N}} = \sqrt{\alpha}|e_0\rangle\langle e_0|$,

$$\alpha|e_0\rangle\langle e_0| = [\mathbf{Q}(X)P_{\mathbb{N}}]^*D^*D[\mathbf{Q}(X)P_{\mathbb{N}}] = |\eta'\rangle\langle \eta'|$$

where $\eta' = P_{\mathbb{N}}Q(X)\eta$ and also $\eta' = z\sqrt{\alpha}e_0$, $z \in \mathbb{T}$. Now $\langle e_m|Q(X)\eta \rangle = \langle e_m|P_{\mathbb{N}}Q(X)\eta \rangle = \langle e_m|\eta' \rangle = 0$ for all $m > 0$ so that $Q(X)\eta = \sum_{n=0}^{\infty} c_n e_{-n}$ for some square summable sequence of complex numbers c_n , i.e. $Q(X)\eta$ is a Hardy function which vanishes on a set $\mathbb{T} \setminus X$ of measure $1 - \ell(X) > 0$. As well known, a Hardy function which vanishes on a set of positive measure is identically zero (see, e.g., [22, Theorem 1]). Therefore, $Q(X)\eta = 0$, $\eta' = 0$, and $\alpha|e_0\rangle\langle e_0| = 0$, yielding $\alpha = 0$. \square

Lemma 2. Let $E \in \mathcal{L}(\mathcal{H})$ be a positive operator such that $\langle n|E|n \rangle = 0$ for all $n > r$ where $r \in \mathbb{N}$, and let $X \in \mathcal{B}(\mathbb{T})$ be such that $\Phi(X) \neq I$. Then $E \leq \Phi(X)$ implies $E = 0$.

Proof. The proof is by induction on r . First we note that, by positivity, if $\langle n|E|n \rangle = 0$ for some n , then $\langle m|E|n \rangle = \overline{\langle n|E|m \rangle} = 0$ for all $m \in \mathbb{N}$. The condition $E \leq \Phi(X)$ implies

$$\langle r|E|r \rangle |0\rangle\langle 0| = WEW^* \leq W\Phi(X)W^* = \Phi(X)$$

where $W = \sum_{k=0}^{\infty} |k\rangle\langle k+r|$. From Lemma 1 one gets $\langle r|E|r \rangle = 0$ and by induction $\langle n|E|n \rangle = 0$ for all $n \in \mathbb{N}$, i.e. $E = 0$. \square

Theorem 1. For any finite subset Y of \mathbb{N} and $X \in \mathcal{B}(\mathbb{T})$ such that $\Phi(X) \neq I$,

$$\Phi(X) \wedge \mathbf{N}(Y) = 0.$$

Proof. Clearly, the claim holds if $Y = \emptyset$ (i.e. $\mathbf{N}(Y) = 0$) so that we assume that Y is finite and non-empty. Assume that there is an effect E such that $E \leq \Phi(X)$ and $E \leq \mathbf{N}(Y)$. Thus, $r = \max Y \in \mathbb{N}$, $\mathbf{N}(Y) \leq R = \sum_{n=0}^r |n\rangle\langle n|$, $\langle n|E|n \rangle \leq \langle n|R|n \rangle = 0$ for all $n > r$. Since also $E \leq \Phi(X)$, Lemma 2 now implies that $E = 0$, that is, 0 is the only lower bound of $\Phi(X)$ and $\mathbf{N}(Y)$. \square

We note that (3.1) is equivalent with the seemingly weaker requirement that this condition holds for all singletons $Y = \{n\}$. Finally, we give bounds for the joint predictability of number and phase.

Corollary 1. For any $X \in \mathcal{B}(\mathbb{T})$, with $\ell(X) < 1$, and for any finite $Y \subset \mathbb{N}$,

$$\sup_{\rho \in \mathcal{S}(\mathcal{H})} (\Phi_{\rho}(X) + \mathbf{N}_{\rho}(Y)) \leq 1 + \sqrt{a_+} < 2,$$

where a_+ is the largest eigenvalue the (finite rank) operator $\mathbf{N}(Y)\Phi(X)\mathbf{N}(Y)$.

Proof. Considering Φ and \mathbf{N} as the Naimark projections of \mathbf{Q} and \mathbf{P} on the subspace $P_{\mathbf{N}}(\tilde{\mathcal{H}})$ of $L^2(\mathbb{T})$, we have

$$\sup_{\rho \in \mathcal{S}(\mathcal{H})} (\Phi_{\rho}(X) + \mathbf{N}_{\rho}(Y)) \leq \sup_{\rho \in \mathcal{S}(\tilde{\mathcal{H}})} (\mathbf{Q}_{\rho}(X) + \mathbf{P}_{\rho}(Y)).$$

Using the results of [9] the numerical range $\{(\langle f|\mathbf{P}(Y)f\rangle, \langle f|\mathbf{Q}(X)f\rangle) \mid f \in \tilde{\mathcal{H}}, \|f\| = 1\}$ of the pair of projections $\mathbf{P}(Y), \mathbf{Q}(X)$ can completely be determined. Since $\mathbf{P}(Y) \wedge \mathbf{Q}(X) = 0$, the point $(1, 1)$ is now excluded from this range. It suffice to recall here that the numerical range is a convex subset of $[0, 1] \times [0, 1]$ [9, Proposition 1] and that for any unit vector $f \in L^2(\mathbb{T})$, the sum $\langle f|\mathbf{Q}(X)f\rangle + \langle f|\mathbf{P}(Y)f\rangle$ is bounded by the number $1 + \sqrt{a_+}$, where a_+ is the maximal eigenvalue of the positive finite rank operator $\mathbf{P}(Y)\mathbf{Q}(X)\mathbf{P}(Y)$ [9, Proposition 5]. Note that the spectra of the operators $\mathbf{N}(Y)\Phi(X)\mathbf{N}(Y)$ and $\mathbf{P}(Y)\mathbf{Q}(X)\mathbf{P}(Y)$ are identical. Since $\text{tr}[\rho\Phi(X)] < 1$ for any state $\rho \in \mathcal{S}(\mathcal{H})$ (see, for instance, [7, Proposition 16.2]), the eigenvalue a_+ is strictly less than one. \square

4. ERRORS IN APPROXIMATE JOINT MEASUREMENTS OF \mathbf{N} AND Φ

We study next the necessary errors appearing in an approximate joint measurement of number and canonical phase. We follow the idea, expounded, for instance, in [10, pp. 197-8], that “measurement error” is to be found by comparing a “real” measurement outcome statistics with the desired one. We take this to mean the comparison of the actual measurement outcome distributions with the ideal ones. Such a comparison can be based on various methods. Here we follow the approach initiated in [11] and further developed in [12, 13] where the error is quantified using the Wasserstein distance between probability measures. For simplicity, we use only the Wasserstein-2 distances and fix the metrics to be the arc distance on \mathbb{T} , $d(\theta, \theta') = \min_{n \in \mathbb{Z}} |\theta - \theta' - 2\pi n|$, and the standard distance on \mathbb{N} , $d(m, n) = |m - n|$.

Let $\mathbf{M}_1 : \mathcal{B}(\mathbb{T}) \rightarrow \mathcal{L}(\mathcal{H})$ and $\mathbf{M}_2 : \mathcal{B}(\mathbb{N}) \rightarrow \mathcal{L}(\mathcal{H})$ be any two observables (semispectral measures) which approximate measurements of Φ and \mathbf{N} , respectively. The error in approximating Φ by \mathbf{M}_1 is now defined as

$$(4.1) \quad d(\mathbf{M}_1, \Phi) = \sup_{\rho} D((\mathbf{M}_1)_{\rho}, \Phi_{\rho}),$$

where $D((\mathbf{M}_1)_{\rho}, \Phi_{\rho})$ is the Wasserstein-2 distance between the probability measures $(\mathbf{M}_1)_{\rho}$ and Φ_{ρ} , that is,

$$D((\mathbf{M}_1)_{\rho}, \Phi_{\rho}) = \inf_{\gamma} \sqrt{\int_{\mathbb{T} \times \mathbb{T}} d(\theta, \theta')^2 d\gamma(\theta, \theta')},$$

where the infimum is taken over all couplings (joint probabilities) $\gamma : \mathcal{B}(\mathbb{T} \times \mathbb{T}) \rightarrow [0, 1]$ of $(\mathbf{M}_1)_\rho$ and Φ_ρ . Similarly, one defines the error $d(\mathbf{M}_2, \mathbf{N})$. Actually, the existence of a minimizing coupling is known [23, Theorem 4.1].

Remark 2. Canonical phase Φ is not a spectral measure. Still, as pointed out above, it resembles a spectral measure in many respects. In particular, the notion of calibration error

$$d^c(\mathbf{M}_1, \Phi) = \limsup_{\epsilon \rightarrow 0} \{D((\mathbf{M}_1)_\rho, \delta_x) \mid D(\Phi_\rho, \delta_x) \leq \epsilon\}$$

makes sense, along with all spectral measure observables, also to canonical phase and one has $d^c(\mathbf{M}_1, \Phi) \leq d(\mathbf{M}_1, \Phi)$. Moreover, if \mathbf{M}_1 is a smearing of Φ in the sense of a convolution, that is, $\mathbf{M}_1 = \mu * \Phi$ for a probability measure μ , then $d^c(\mathbf{M}_1, \Phi)^2 = d(\mathbf{M}_1, \Phi)^2 = \int_{\mathbb{T}} d(\theta, 0)^2 d\mu = \int_{\mathbb{T}} \min_{n \in \mathbb{Z}} |\theta - 2\pi n|^2 d\mu = \int_{-\pi}^{\pi} \theta^2 d\mu \equiv \mu[2]$. Similarly, if $\mathbf{M}_2 = \nu * \mathbf{N}$ for some probability measure ν , then $d^c(\mathbf{M}_2, \mathbf{N})^2 = d(\mathbf{M}_2, \mathbf{N})^2 = \sum_{k=0}^{\infty} d(k, 0)^2 \nu(\{k\}) = \sum_{k=0}^{\infty} k^2 \nu(\{k\}) \equiv \nu[2]$ [12, Lemmas 7, 11].

For an approximate joint measurement of Φ and \mathbf{N} , the approximators \mathbf{M}_1 and \mathbf{M}_2 must be compatible, that is, margins of a joint observable $\mathbf{M} : \mathcal{B}(\mathbb{T} \times \mathbb{N}) \rightarrow \mathcal{L}(\mathcal{H})$.³ The basic problem is thus to characterize the joint measurement error set

$$(4.2) \quad \mathbf{MU}(\mathbb{T} \times \mathbb{N}) = \{(d(\mathbf{M}_1, \Phi), d(\mathbf{M}_2, \mathbf{N})) \mid \mathbf{M} : \mathcal{B}(\mathbb{T} \times \mathbb{N}) \rightarrow \mathcal{L}(\mathcal{H})\},$$

where \mathbf{M}_j are the cartesian margins of \mathbf{M} . Here we use the notation $\mathbf{MU}(\mathbb{T} \times \mathbb{N})$ to indicate explicitly the value space of the approximate joint observables.

The incompatibility of Φ and \mathbf{N} implies that the point $(0, 0)$ is not in the set $\mathbf{MU}(\mathbb{T} \times \mathbb{N})$. On the other hand, if one of the errors is zero, then \mathbf{M} is a smearing of the exact margin \mathbf{M}_1 or \mathbf{M}_2 . From the below Proposition 1 we then conclude that if $d(\mathbf{M}_1, \Phi) = 0$, that is, $\mathbf{M}_1 = \Phi$, then $d(\mathbf{M}_2, \mathbf{N})$ cannot be finite. On the other hand, if $\mathbf{M}_2 = \mathbf{N}$, then $\pi/\sqrt{3} \leq d(\mathbf{M}_1, \Phi) \leq \pi$ where the lower bound is attained with the kernel $p_k = \ell$, $k \in \mathbb{N}$, and the upper bound with $p_k = \delta_\beta$, $k \in \mathbb{N}$, where $\beta \in [0, 2\pi)$.

The semigroup structure of the outcome space of the number measurements has thwarted our attempts to determine directly the set (4.2). However, we can still bound this set by enlarging the joint values set $\mathbb{T} \times \mathbb{N}$ to $\mathbb{T} \times \mathbb{Z}$, that is, studying instead of (4.2) the set $\mathbf{MU}(\mathbb{T} \times \mathbb{Z})$. This case reduces to the case of position \mathbf{Q} and momentum \mathbf{P} (or angle and (\mathbb{Z}) -number) on $\tilde{\mathcal{H}} = L^2(\mathbb{T})$ studied in great detail in [14].

³See [7, Theorem 11.1] for several alternative definitions.

Let $G^\sigma : \mathcal{B}(\mathbb{T} \times \mathbb{Z}) \rightarrow \mathcal{L}(\tilde{\mathcal{H}})$ be the covariant phase space observable generated by a state $\sigma \in \mathcal{S}(\tilde{\mathcal{H}})$ so that its margins are the smeared position and momentum observables $Q_\sigma * Q$ and $P_\sigma * P$, smeared by the position and momentum distributions Q_σ and P_σ in state σ , respectively [25, 14]. The observable $E^\sigma : \mathcal{B}(\mathbb{T} \times \mathbb{Z}) \rightarrow \mathcal{L}(\mathcal{H})$, defined as

$$(4.3) \quad E^\sigma(X \times Y) = V^* G^\sigma(X \times Y) V,$$

has then the smeared phase $E_1^\sigma = Q_\sigma * \Phi$ and smeared number $E_2^\sigma = P_\sigma * N$ as its margins. By Remark 2, the errors now reduce to the preparation uncertainties of Q and P in state σ

$$d(E_1^\sigma, \Phi) = \sqrt{Q_\sigma[2]} \quad \text{and} \quad d(E_2^\sigma, N) = \sqrt{P_\sigma[2]}.$$

The following proposition bounds the error set $\text{MU}(\mathbb{T} \times \mathbb{N})$ by the bounds of the larger set $\text{MU}(\mathbb{T} \times \mathbb{Z})$.

Proposition 1. Let $F : \mathcal{B}(\mathbb{T} \times \mathbb{Z}) \rightarrow \mathcal{L}(\mathcal{H})$ be an observable such that $d(F_2, N) < \infty$. Then there exists a state operator σ on $\tilde{\mathcal{H}}$, such that

$$d(E_1^\sigma, \Phi) \leq d(F_1, \Phi) \quad \text{and} \quad d(E_2^\sigma, N) \leq d(F_2, N),$$

where E^σ is given by (4.3). In particular, the boundary curve for the error set $\text{MU}(\mathbb{T} \times \mathbb{Z})$, which includes the set $\text{MU}(\mathbb{T} \times \mathbb{N})$, is the same as for Q and P on $\tilde{\mathcal{H}}$, as characterised in [14].

The idea behind the proof is the following:

- (1) Starting from F , construct an observable M on $\tilde{\mathcal{H}}$ in such a way that the errors of its margins with respect to Q and P reflect the original errors.
- (2) Average M with respect to phase space translations so that the errors (actually, the state dependent errors) do not increase.
- (3) Project the averaged observable \bar{M} back to \mathcal{H} to get the desired result.

Proof. Let $F : \mathcal{B}(\mathbb{T} \times \mathbb{Z}) \rightarrow \mathcal{L}(\mathcal{H})$ be an observable with $d(F_2, N) < \infty$. Define an observable $M : \mathcal{B}(\mathbb{T} \times \mathbb{Z}) \rightarrow \mathcal{L}(\tilde{\mathcal{H}})$ via

$$(4.4) \quad M(X \times Y) = V F(X \times Y) V^* + \sum_{n=1}^{\infty} \ell(X) \langle n | F_2(-Y) | n \rangle |e_{-n}\rangle \langle e_{-n}|.$$

We now proceed by calculating the error $d(M_2, P)$ for the second margin M_2 . By Remark 2, it is sufficient to take the supremum over the eigenstates $|e_k\rangle$ of P , and we have the probabilities

$$\mathbf{p}_{e_k}^{M_2}(Y) = \langle e_k | M(\mathbb{T} \times Y) | e_k \rangle = \begin{cases} \langle k | F_2(Y) | k \rangle & \text{for } k \geq 0, \\ \langle -k | F_2(-Y) | -k \rangle & \text{for } k < 0. \end{cases}$$

Since $\mathbf{p}_{e_k}^P = \delta_k$, we have

$$d(\mathbf{p}_{e_k}^{M_2}, \mathbf{p}_{e_k}^P) = \left(\sum_{l=-\infty}^{\infty} |l - k|^2 \mathbf{p}_{e_k}^{M_2}(\{l\}) \right)^{1/2}$$

so that for $k \geq 0$,

$$d(\mathbf{p}_{e_k}^{M_2}, \mathbf{p}_{e_k}^P) = \left(\sum_{l=-\infty}^{\infty} |l - k|^2 \langle k | F_2(\{l\}) | k \rangle \right)^{1/2} = d(\mathbf{p}_{|k\rangle}^{F_2}, \mathbf{p}_{|k\rangle}^N)$$

whereas for $k < 0$ we have

$$\begin{aligned} d(\mathbf{p}_{e_k}^{M_2}, \mathbf{p}_{e_k}^P) &= \left(\sum_{l=-\infty}^{\infty} |l - k|^2 \langle -k | F_2(\{-l\}) | -k \rangle \right)^{1/2} \\ &= \left(\sum_{l=-\infty}^{\infty} |l - (-k)|^2 \langle -k | F_2(\{l\}) | -k \rangle \right)^{1/2} \\ &= d(\mathbf{p}_{|-k\rangle}^{F_2}, \mathbf{p}_{|-k\rangle}^N) \end{aligned}$$

Since $d(F_2, N)$ is also obtained by calculating the supremum over the number states $|k\rangle$, we have that

$$(4.5) \quad d(M_2, P) = \sup_{k \in \mathbb{Z}} d(\mathbf{p}_{e_k}^{M_2}, \mathbf{p}_{e_k}^P) = \sup_{k \in \mathbb{N}} d(\mathbf{p}_{|k\rangle}^{F_2}, \mathbf{p}_{|k\rangle}^N) = d(F_2, N).$$

For the first margin, we do not get such an equality due to the trivial term coming from the last term in Eq. (4.4). However, we may restrict to the states

$$\mathcal{S}_+(\tilde{\mathcal{H}}) = \{\rho \in \mathcal{S}(\tilde{\mathcal{H}}) \mid \langle e_k | \rho e_l \rangle = 0 \text{ for all } k < 0 \text{ or } l < 0\}$$

so that $V^* \mathcal{S}_+(\tilde{\mathcal{H}}) V = \mathcal{S}(\mathcal{H})$. Since for any $\rho \in \mathcal{S}_+(\tilde{\mathcal{H}})$ we have $\text{tr}[\rho M_1(Y)] = \text{tr}[V^* \rho V F_1(X)]$ and $\text{tr}[\rho Q(X)] = \text{tr}[V V^* \rho V V^* Q(X)] = \text{tr}[V^* \rho V \Phi(X)]$, we have, in particular, that

$$(4.6) \quad d(F_1, \Phi) = \sup_{\rho \in \mathcal{S}(\mathcal{H})} d(\mathbf{p}_\rho^{F_1}, \mathbf{p}_\rho^\Phi) = \sup_{\rho \in \mathcal{S}_+(\tilde{\mathcal{H}})} d(\mathbf{p}_{V^* \rho V}^{F_1}, \mathbf{p}_{V^* \rho V}^\Phi) = \sup_{\rho \in \mathcal{S}_+(\tilde{\mathcal{H}})} d(\mathbf{p}_\rho^{M_1}, \mathbf{p}_\rho^Q).$$

The next step is to average the observable M with respect to phase space translations, and to show that the averaged observable \overline{M} satisfies

$$(4.7) \quad \sup_{\rho \in \mathcal{S}_+(\tilde{\mathcal{H}})} d(\mathbf{p}_\rho^{\overline{M}_1}, \mathbf{p}_\rho^Q) = \sup_{\rho \in \mathcal{S}_+(\tilde{\mathcal{H}})} d(\mathbf{p}_\rho^{M_1}, \mathbf{p}_\rho^Q) \quad \text{and} \quad d(\overline{M}_2, P) = d(M_2, P)$$

We perform the averaging by using an invariant mean m on $\mathbb{T} \times \mathbb{Z}$, see, for instance, [24]. For any trace class operator $T \in \mathcal{T}(\tilde{\mathcal{H}})$ and any bounded continuous function $f : \mathbb{T} \times \mathbb{Z} \rightarrow \mathbb{C}$, define

$$\Theta[T, f](\theta, k) = \text{tr} [T W(\theta, k)^* M(f^{(\theta, k)}) W(\theta, k)]$$

where $W(\theta, k)$ are the Weyl operators and $f^{(\theta, k)}$ denotes the translate of f . Then $\Theta[T, f] : \mathbb{T} \times \mathbb{Z} \rightarrow \mathbb{C}$ is a bounded continuous function, and by standard arguments the formula

$$\mathrm{tr} [T\overline{\mathbf{M}}(f)] = m(\Theta[T, f])$$

determines a covariant phase space observable $\overline{\mathbf{M}} : \mathcal{B}(\mathbb{T} \times \mathbb{Z}) \rightarrow \mathcal{L}(\widetilde{\mathcal{H}})$ (since $d(\mathbf{M}_2, \mathbf{P}) = d(\mathbf{F}_2, \mathbf{N}) < \infty$ and $d(\mathbf{M}_1, \Phi) < \infty$ trivially by the compactness of \mathbb{T} , the normalization of $\overline{\mathbf{M}}$ is guaranteed [11]).

Let $\rho \in \mathcal{S}(\widetilde{\mathcal{H}})$. Then by the Kantorovich duality, for any bounded continuous functions $f, g : \mathbb{T} \rightarrow \mathbb{R}$ such that $f(\theta) - g(\theta') \leq d(\theta, \theta')^2$ we have

$$\mathrm{tr} [\rho(\mathbf{M}_1(f) - \mathbf{Q}(g))] \leq d(\mathbf{p}_\rho^{\mathbf{M}_1}, \mathbf{p}_\rho^{\mathbf{Q}}).$$

Since the above class of functions is invariant with respect to translations, we have

$$\begin{aligned} \mathrm{tr} [W(\theta, k)\rho W(\theta, k)^*(\mathbf{M}_1(f^{(\theta)}) - \mathbf{Q}(g^{(\theta)}))] &= \mathrm{tr} [\rho W(\theta, k)^*\mathbf{M}_1(f^{(\theta)})W(\theta, k)] - \mathrm{tr} [\rho\mathbf{Q}(g)] \\ &\leq d(\mathbf{p}_\rho^{\mathbf{M}_1}, \mathbf{p}_\rho^{\mathbf{Q}}), \end{aligned}$$

or equivalently,

$$\mathrm{tr} [\rho W(\theta, k)^*\mathbf{M}(f_1^{(\theta, k)})W(\theta, k)] \leq \mathrm{tr} [\rho\mathbf{Q}(g)] + d(\mathbf{p}_\rho^{\mathbf{M}_1}, \mathbf{p}_\rho^{\mathbf{Q}})$$

where $f_1(\alpha, l) = f(\alpha)$. By applying the invariant mean, we obtain

$$\mathrm{tr} [\rho\overline{\mathbf{M}}_1(f)] - \mathrm{tr} [\rho\mathbf{Q}(g)] \leq d(\mathbf{p}_\rho^{\mathbf{M}_1}, \mathbf{p}_\rho^{\mathbf{Q}})$$

for all f, g . By taking the supremum over such functions we get

$$d(\mathbf{p}_\rho^{\overline{\mathbf{M}}_1}, \mathbf{p}_\rho^{\mathbf{Q}}) \leq d(\mathbf{p}_\rho^{\mathbf{M}_1}, \mathbf{p}_\rho^{\mathbf{Q}})$$

for all $\rho \in \mathcal{S}(\widetilde{\mathcal{H}})$. The same holds also for the second margin. Hence, we conclude that Eq. (4.7) holds.

Since $\overline{\mathbf{M}}$ is a covariant phase space observable, we know that $\overline{\mathbf{M}} = \mathbf{G}^\sigma$ for some $\sigma \in \mathcal{S}(\widetilde{\mathcal{H}})$. We now set $\mathbf{E}^\sigma = V^*\mathbf{G}^\sigma V = V^*\overline{\mathbf{M}}V$, so that

$$\begin{aligned} d(\mathbf{E}_1^\sigma, \Phi) &= d(V^*\overline{\mathbf{M}}_1 V, V^*\mathbf{Q}V) = \sup_{\rho \in \mathcal{S}(L^2(\mathbb{R}))} d(\mathbf{p}_{V\rho V^*}^{\overline{\mathbf{M}}_1}, \mathbf{p}_{V\rho V^*}^{\mathbf{Q}}) = \sup_{\rho \in \mathcal{S}_+(\widetilde{\mathcal{H}})} d(\mathbf{p}_\rho^{\overline{\mathbf{M}}_1}, \mathbf{p}_\rho^{\mathbf{Q}}) \\ &\leq \sup_{\rho \in \mathcal{S}_+(\widetilde{\mathcal{H}})} d(\mathbf{p}_\rho^{\mathbf{M}_1}, \mathbf{p}_\rho^{\mathbf{Q}}) = d(\mathbf{F}_1, \Phi) \end{aligned}$$

and similarly $d(\mathbf{E}_2^\sigma, \mathbf{N}) \leq d(\mathbf{F}_2, \mathbf{N})$.

□

For any \mathbf{F} for which $d(\mathbf{F}_2, \mathbf{N})$ is finite there is thus an \mathbf{E}^σ such that $d(\mathbf{E}_1^\sigma, \Phi) \leq d(\mathbf{F}_1, \Phi)$ and $d(\mathbf{E}_2^\sigma, \mathbf{N}) \leq d(\mathbf{F}_2, \mathbf{N})$, so that⁴

$$d(\mathbf{F}_1, \Phi)^2 + d(\mathbf{F}_2, \mathbf{N})^2 \geq d(\mathbf{E}_1^\sigma, \Phi)^2 + d(\mathbf{E}_2^\sigma, \mathbf{N})^2 = \mathbf{Q}_\sigma[2] + \mathbf{P}_\sigma[2] = \text{tr} [\sigma(Q^2 + P^2)] \geq \tilde{E}_0,$$

where $\tilde{E}_0 > 0$ is the smallest eigenvalue of the oscillator energy operator $Q^2 + P^2$ in $\tilde{\mathcal{H}}$. Though the existence of \tilde{E}_0 is known, we can only give its approximate value $\tilde{E}_0 \approx 0.9996$ (see Appendix A). If $\psi \in \tilde{\mathcal{H}}$ is a corresponding eigenvector then $\mathbf{E}^{|\psi\rangle\langle\psi|}$ is an optimal joint measurement of Φ and \mathbf{N} with the value space $\mathbb{T} \times \mathbb{Z}$. For a detailed analysis of the boundary curve of the convex hull of the monotone hull of the error sets $\mathbf{MU}(\mathbb{T} \times \mathbb{Z})$ we refer to [14], in particular, its Sections IV, V, and VI.

Remark 3. By extending the value space of the approximate joint measurements from $\mathbb{T} \times \mathbb{N}$ to $\mathbb{T} \times \mathbb{Z}$, we are potentially enlarging also the initial error set. This leaves us with a question if the inclusion $\mathbf{MU}(\mathbb{T} \times \mathbb{N}) \subseteq \mathbf{MU}(\mathbb{T} \times \mathbb{Z})$ is a proper one. Natural candidates for optimal joint observables on $\mathbb{T} \times \mathbb{N}$ are the observables \mathbf{E}^σ whose support is contained in $\mathbb{T} \times \mathbb{N}$. This amounts to the requirement that the generating operator $\sigma \in \mathcal{S}(\tilde{\mathcal{H}})$ is supported on the positive number states, that is, $\langle e_k | \sigma e_l \rangle = 0$ wherever $k < 0$ or $l < 0$. Optimizing over such states is equivalent to optimizing the preparation uncertainties for Φ and \mathbf{N} over all states $\rho \in \mathcal{S}(\mathcal{H})$. Based on numerical calculations, the uncertainties lead to a strict subset of $\mathbf{MU}(\mathbb{T} \times \mathbb{Z})$ giving evidence that this inclusion could be a proper one. However, we are lacking an argument which would show that these are indeed optimal $\mathbb{T} \times \mathbb{N}$ valued approximate joint observables. We are thus also left with the problem of proving or disproving that the optimal $\mathbb{T} \times \mathbb{N}$ valued approximate joint observables for Φ and \mathbf{N} are given by those \mathbf{E}^σ whose support is contained in $\mathbb{T} \times \mathbb{N}$.

ACKNOWLEDGMENTS

JS acknowledges financial support from the EU through the Collaborative Projects QuProCS (Grant Agreement No. 641277).

APPENDIX A. PROOF OF THE EXISTENCE OF THE EIGENVALUE

In this appendix we give a simple proof of the well-known fact that the operator $P^2 + Q^2$ in $\tilde{\mathcal{H}}$, as well as the operator $N^2 + \Phi[2]$ in \mathcal{H} , has a discrete spectrum with a strictly positive lowest eigenvalue. For that end, we fix a separable Hilbert space (with the identity I) and

⁴Recall that due to the arc distance on \mathbb{T} , the error $\mathbf{Q}_\sigma[2] = \int_{-\pi}^{\pi} \theta^2 d\mathbf{Q}_\sigma(\theta)$ so that also the operator $Q^2 = \int_{-\pi}^{\pi} \theta^2 d\mathbf{Q}(\theta)$.

assume that all operators (bounded or not) act in this space. We let \mathcal{B} denote the unit ball of the Hilbert space.

Lemma 3. Let E and F be bounded operators such that $0 \leq E \leq F \leq I$ and $\|I - E\| < 1$. Then E and F are invertible and $E^{-1} \geq F^{-1} \geq I$.

Proof. Since $\|I - E\| < 1$ it follows that $\lim_{s \rightarrow \infty} \|I - E\|^s = 0$, and $I + \sum_{k=1}^{\infty} (I - E)^k$ converges in the operator norm to a bounded operator. Moreover,

$$\underbrace{E}_{I - (I - E)} \left[I + \sum_{k=1}^{s-1} (I - E)^k \right] = I - (I - E)^s \rightarrow I$$

when $s \rightarrow \infty$, so that

$$E^{-1} = I + \sum_{k=1}^{\infty} (I - E)^k \geq I.$$

Indeed, $(I - E)^k = \int_0^{\|I - E\|} x^k d\mathbf{M}(x) \geq 0$, for all $k = 1, 2, \dots$, where \mathbf{M} is the spectral measure of $I - E \geq 0$. Since $0 \leq I - F \leq I - E$ it follows that $\|I - F\| = \sup_{\psi \in \mathcal{B}} \langle \psi | (I - F) \psi \rangle \leq \|I - E\| < 1$, and (similarly as above) one sees that F is invertible. Let $F^{1/2}$ (resp. $F^{-1/2}$) be the square root operators of F (resp. $F^{-1} \geq I \geq 0$). Now $G = F^{-1/2} E F^{-1/2} \geq 0$ is invertible with the inverse $G^{-1} = F^{1/2} E^{-1} F^{1/2}$ and the condition $E \leq F$ is equivalent to $G \leq I$. Now $\|I - G\| < 1$ since otherwise (i.e. if $\|I - G\| = 1$) there would exist a sequence $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{B}$ of unit vectors such that $\lim_{n \rightarrow \infty} \langle \psi_n | (I - G) \psi_n \rangle = 1$, that is, $\|G^{1/2} \psi_n\|^2 = \langle \psi_n | G \psi_n \rangle \rightarrow 0$, $n \rightarrow \infty$, and thus $1 = \|\psi_n\| = \|G^{-1/2} G^{1/2} \psi_n\| \leq \|G^{-1/2}\| \|G^{1/2} \psi_n\| \rightarrow 0$ when $n \rightarrow \infty$. Hence, by the above calculation, $G^{-1} \geq I$ so that $E^{-1} = F^{-1/2} G^{-1} F^{-1/2} \geq F^{-1}$. \square

Proposition 2. Let T be a positive (possibly unbounded) selfadjoint operator with a purely discrete non-degenerate spectrum. Assume that its eigenvalues $0 \leq p_0 < p_1 < p_2 < \dots$ are such that $\sum_n (1 + p_n)^{-1} < \infty$. Let V be a positive bounded operator. Then the spectrum of $H = T + V$ is discrete. The lowest eigenvalue of H is zero if and only if $p_0 = 0$ and $V\phi_0 = 0$ where $\phi_0 \neq 0$ is an eigenvector of T related to the eigenvalue p_0 .

Proof. If the Hilbert space is finite dimensional then the proof is trivial so we consider only an infinite dimensional case. By assumption, $T = \sum_{n=0}^{\infty} p_n |\phi_n\rangle \langle \phi_n|$ for an orthonormal basis $\{\phi_n\}$. The domain of T is $\mathcal{D} = \left\{ \sum_{n=0}^{\infty} c_n \phi_n \mid \sum_{n=0}^{\infty} p_n^2 |c_n|^2 < \infty \right\}$. Now $(T + I)^{-1} = \sum_{n=0}^{\infty} p'_n |\phi_n\rangle \langle \phi_n|$, with $p'_n = (1 + p_n)^{-1} \in (0, 1]$, is a positive trace class operator. Define $W = T + \|V\| I + I$ on \mathcal{D} so that

$$W^{-1/2} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{p_n + \|V\| + 1}} |\phi_n\rangle \langle \phi_n|$$

is a bounded operator with the norm $\|W^{-1/2}\| = \sup_n (p_n + \|V\| + 1)^{-1/2} = (p_0 + \|V\| + 1)^{-1/2}$. Let $A = T + I$ and $B = T + V + I$ be positive operators defined on \mathcal{D} . Since $V \leq \|V\|I$ one gets $0 \leq \langle \psi | A \psi \rangle \leq \langle \psi | B \psi \rangle \leq \langle \psi | W \psi \rangle$, $\psi \in \mathcal{V} = \text{lin}\{\phi_n\} \subset \mathcal{D}$, or, since $W^{-1/2}\mathcal{V} \subset \mathcal{V}$,

$$0 \leq W^{-1/2} A W^{-1/2} \leq W^{-1/2} B W^{-1/2} \leq I$$

where, e.g. $W^{-1/2} B W^{-1/2}$ is a bounded operator determined uniquely by the corresponding bounded sesquilinear form $\mathcal{V} \times \mathcal{V} \ni (\varphi, \psi) \mapsto \langle W^{-1/2} \varphi | B W^{-1/2} \psi \rangle \in \mathbb{C}$.

Since $\|I - W^{-1/2} A W^{-1/2}\| = \sup_n \left(\frac{\|V\|}{p_n + \|V\| + 1} \right) = \frac{\|V\|}{p_0 + \|V\| + 1} < 1$, from Lemma 3, one sees that

$$(W^{-1/2} A W^{-1/2})^{-1} \geq (W^{-1/2} B W^{-1/2})^{-1} \geq I,$$

that is, $p'_n = \langle \phi_n | (T + I)^{-1} \phi_n \rangle \geq \langle \phi_n | (T + V + I)^{-1} \phi_n \rangle \geq (p_n + \|V\| + 1)^{-1} > 0$ and

$$\sum_{n=0}^{\infty} \langle \phi_n | W^{-1} \phi_n \rangle \leq \sum_{n=0}^{\infty} \langle \phi_n | (T + V + I)^{-1} \phi_n \rangle \leq \sum_{n=0}^{\infty} p'_n < \infty$$

showing that $(T + V + I)^{-1} \geq W^{-1}$ is a (positive) trace-class operator. Let

$$(T + V + I)^{-1} = \sum_{l=0}^{\infty} \lambda_l |\varphi_l\rangle \langle \varphi_l|$$

where $\{\varphi_l\}$ is an orthonormal basis and $\lambda_l \in (0, 1]$, $\sum_{l=0}^{\infty} \lambda_l < \infty$. Hence,

$$H = T + V = \sum_{l=0}^{\infty} q_l |\varphi_l\rangle \langle \varphi_l|$$

where $q_l = \lambda_l^{-1} - 1 \geq 0$. Finally, let $\phi \in \mathcal{D}$. Then, $H\phi = 0$ if and only if $0 = \langle \phi | H \phi \rangle = \langle \phi | T \phi \rangle + \langle \phi | V \phi \rangle$ if and only if $\langle \phi | T \phi \rangle = 0 = \langle \phi | V \phi \rangle$ if and only if $T\phi = 0 = V\phi$. \square

Note that, in the context of the above Proposition, all operators $T + cV$, $c > 0$, have discrete spectra, and their spectra have non-zero smallest eigenvalues (i.e. positive spectra) if $T + V$ has a positive spectrum.

In either case, $\tilde{H} = P^2 + Q^2 = P^2 + \int_{-\pi}^{\pi} \theta^2 dQ(\theta)$ (in $\tilde{\mathcal{H}}$) or $H = N^2 + \Phi[2] = N^2 + \int_{-\pi}^{\pi} \theta^2 d\Phi(\theta)$ (in \mathcal{H}), the assumptions of Proposition 2 are satisfied; in particular, both of the positive operators Q^2 or $\Phi[2]$ have a purely continuous spectrum (with no eigenvalues): $\sigma(Q^2) = \sigma(\Phi[2]) = [0, \pi^2]$. Hence both operators \tilde{H} , H have strictly positive lowest eigenvalues \tilde{E}_0 , E_0 , respectively. Also, this follows directly from Proposition 2 by noting that $\langle e_0 | Q^2 e_0 \rangle = \langle 0 | \Phi[2] | 0 \rangle = \int_{-\pi}^{\pi} \theta^2 d\theta / (2\pi) > 0$, i.e. $P^2 e_0 = 0$ but $Q^2 e_0 \neq 0$ and $N^2 |0\rangle = 0$ but $\Phi[2] |0\rangle \neq 0$. Numerically, $\tilde{E}_0 \approx 0.9996\dots$ associated with the (normalized) eigenvector $\tilde{\psi}_{\min} = \sum_{s=-\infty}^{\infty} c_s e_s$ where $c_0 \approx 0.7518$, $c_{\pm 1} \approx 0.4550$, $c_{\pm 2} \approx 0.1017$, $c_{\pm 3} \approx 0.0083$, $c_{\pm 4} \approx 0.0002$, etc. Moreover, $E_0 \approx 1.5818\dots$ with the eigenvector $\psi_{\min} \approx 0.7276|0\rangle + 0.6632|1\rangle + 0.1745|2\rangle + 0.0167|3\rangle + 0.0002|4\rangle + \dots$. To

conclude, if $M : \mathcal{B}(\mathbb{T} \times \mathbb{Z}) \rightarrow \mathcal{L}(\mathcal{H})$ is any approximate joint measurement of Φ and N , with $d(M_2, N) < \infty$, then

$$d(M_1, \Phi) + d(M_2, N) \geq \tilde{E}_0 \approx 1.$$

It remains, however, an open question if the eigenvalue E_0 of $N^2 + \Phi[2]$ bounds the error sum $d(M_1, \Phi) + d(M_2, N)$ for the $\mathbb{T} \times \mathbb{N}$ -valued approximate joint measurements of phase and number.

Remark 4. The above numerical results for the smallest eigenvalues and the corresponding eigenvectors is based on the following facts: Let $H = T + V$, $T = \sum_{n=0}^{\infty} p_n |\phi_n\rangle\langle\phi_n|$, be as in Proposition 2 (we assume that the Hilbert space is infinite-dimensional). Let $C_{\psi_{\min}} \geq 0$ be the lowest eigenvalue of H with the (normalized) eigenvector ψ_{\min} . Let $P_k = \sum_{n=0}^k |\phi_n\rangle\langle\phi_n|$ so that $P_k \rightarrow I$, $k \rightarrow \infty$, with respect to the strong (and weak) operator topology. Denote $H_k = P_k H P_k \geq 0$ and let α_k be the smallest eigenvalue of the ‘finite positive matrix’ H_k . Let $\eta_k \in \mathcal{B}$, $P_k \eta_k = \eta_k$, be the corresponding eigenvector of H_k , that is, $H_k \eta_k = \alpha_k \eta_k$. Since $\alpha_k = \inf\{\langle\psi|H_k\psi\rangle \mid \psi \in \mathcal{B}, P_k\psi = \psi\}$ and $P_{k+1}P_k = P_k$ one gets

$$C_{\psi_{\min}} \leq \langle\eta_{k+1}|H\eta_{k+1}\rangle = \alpha_{k+1} \leq \alpha_k \leq \langle P_k \psi_{\min} | H_k P_k \psi_{\min} \rangle \|P_k \psi_{\min}\|^{-2}.$$

Since $\lim_{k \rightarrow \infty} \|P_k \psi_{\min}\| = 1$, to get $\lim_{k \rightarrow \infty} \alpha_k = C_{\psi_{\min}}$, one is left to show that (when $k \rightarrow \infty$)

$$\langle P_k \psi_{\min} | H_k P_k \psi_{\min} \rangle = \langle P_k \psi_{\min} | H P_k \psi_{\min} \rangle \rightarrow \langle \psi_{\min} | H \psi_{\min} \rangle = C_{\psi_{\min}}$$

or⁵ that $H P_k \psi_{\min} \rightarrow H \psi_{\min} = C_{\psi_{\min}} \psi_{\min}$. But this is obvious (see the end of the proof of the proposition):

$$\|H \psi_{\min} - H P_k \psi_{\min}\|^2 = \sum_{l=1}^{\infty} (q_l)^2 \underbrace{|\langle \varphi_l | (I - P_k) \psi_{\min} \rangle|^2}_{\rightarrow 0 \ (k \rightarrow \infty)} \rightarrow 0.$$

We have proved that $\lim_{k \rightarrow \infty} \alpha_k = C_{\psi_{\min}}$, i.e. $\lim_{k \rightarrow \infty} \langle \eta_k | H \eta_k \rangle = \langle \psi_{\min} | H \psi_{\min} \rangle$. Hence, one can numerically solve the smallest eigenvalues α_k of the finite matrices H_k . When k is large enough one gets $C_{\psi_{\min}} \approx \alpha_k$.

REFERENCES

- [1] N. Bohr, “The Quantum Postulate and the Recent Development of Atomic Theory,” *Nature* **121**, 580-590 (1928).
- [2] W. Pauli, *General Principles of Quantum Mechanics*, Springer, 1980. The original German text *Wellenmechanik*, 1933.
- [3] W. Heisenberg, “Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik”, *Z. Physik* **43**, 172-198 (1927).

⁵ $\langle \psi | \dots \psi \rangle = \langle P_k \psi | \dots P_k \psi \rangle + \langle P_k^\perp \psi | \dots P_k \psi \rangle + \langle P_k \psi | \dots P_k^\perp \psi \rangle + \langle P_k^\perp \psi | \dots P_k^\perp \psi \rangle$ where $P_k^\perp = I - P_k$

- [4] N. Bohr, “Can Quantum-Mechanical Description of Physical Reality be Considered Complete?”, *Phys. Rev.* **48**, 696-702 (1935).
- [5] K. Kraus, “Complementary observables and uncertainty relations”, *Phys. Rev. D* **35**, 3070-3075 (1987).
- [6] P. Busch, M. Grabowski, P. Lahti, *Operational Quantum Physics*, LNP m 31, Springer, 1995.
- [7] P. Busch, P. Lahti, J.-P. Pellonpää, K. Ylinen, *Quantum Measurement*, Theoretical and Mathematical Physics, Springer International Publishing Switzerland 2016.
- [8] P. Busch, P. Lahti, J.-P. Pellonpää, K. Ylinen, “Are number and phase complementary observables?”, *J. Phys. A: Math. Gen.* **34**, 5923-5935 (2001).
- [9] A. Lenard, “The numerical range of pairs of projections”, *J. Funct. Anal.* **10**, 410-423 (1972).
- [10] G. Ludwig, *Foundations of Quantum Mechanics I*, Springer, 1983.
- [11] R. F. Werner, “The uncertainty relation for joint measurement of position and momentum”, *Quantum Inf. Comput.* **4**, 546-562 (2004).
- [12] P. Busch, P. Lahti, R. F. Werner, “Measurement uncertainty relations”, *J. Math. Phys.* **55**, 042111 (2014).
- [13] P. Busch, P. Lahti, R. F. Werner, “Colloquium: Quantum root-mean-square error and measurement uncertainty relations”, *Rev. Mod. Phys.* **86**, 1261-1281 (2014).
- [14] P. Busch, J. Kiukas, R. F. Werner, “Sharp uncertainty relations for number and angle”, arXiv:1604.00566 [quant-ph].
- [15] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, North Holland Publ. Co., 1982.
- [16] J.-P. Pellonpää, *Covariant Phase Observables in Quantum Mechanics*, Annales Universitatis Turkuensis A **288**, 2002. Available at <http://www.doria.fi/handle/10024/5808>
- [17] T. Heinosaari, J.-P. Pellonpää, “Canonical phase measurement is pure”, *Phys. Rev. A* **80**, 040101(R) (2009).
- [18] J.-P. Pellonpää, J. Schultz, “Measuring the canonical phase with phase-space measurements”, *Phys. Rev. A* **88**, 012121 (2013).
- [19] J.-P. Pellonpää, “On coexistence and joint measurability of rank-1 quantum observables”, *J. Phys. A: Math. Theor.* **47**, 052002 (2014).
- [20] H. Du, C. Deng, Q. Li, “On the infimum problem of Hilbert space effects”, *Science in China: Series A Mathematics* **49**, 545-556 (2006).
- [21] P. J. Lahti and K. Ylinen, “On total noncommutativity in quantum mechanics”, *J. Math. Phys.* **28**, 2614-2617 (1987).
- [22] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, 1964.
- [23] C. Villani, *Optimal Transport: Old and New*, Springer, 2009.
- [24] E. Hewitt, K. A. Ross, *Abstract Harmonic Analysis. Vol. I: Structure of Topological Groups. Integration Theory, Group Representations*, Academic Press, New York, 1963.
- [25] R. F. Werner, “Quantum harmonic analysis on phase space,” *J. Math. Phys.* **25**, 1404 (1984).

TURKU CENTRE FOR QUANTUM PHYSICS, DEPARTMENT OF PHYSICS AND ASTRONOMY, UNIVERSITY OF
TURKU, FI-20014 TURKU, FINLAND

E-mail address: pekka.lahti@utu.fi

TURKU CENTRE FOR QUANTUM PHYSICS, DEPARTMENT OF PHYSICS AND ASTRONOMY, UNIVERSITY OF
TURKU, FI-20014 TURKU, FINLAND

E-mail address: juhpollo@utu.fi

TURKU CENTRE FOR QUANTUM PHYSICS, DEPARTMENT OF PHYSICS AND ASTRONOMY, UNIVERSITY OF
TURKU, FI-20014 TURKU, FINLAND

E-mail address: jussi.schultz@gmail.com